

The Chow ring of the symmetric space
 $Gl(2n, \mathbb{C})/SO(2n, \mathbb{C})$

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March 19, 2001

ABSTRACT. We show this Chow ring is $\mathbb{Z} \oplus \mathbb{Z}$. We do this by partitioning the space into $2n$ subvarieties each of which is fibered over $Gl(2n-2, \mathbb{C})/SO(2n-2, \mathbb{C})$.

1 Introduction

This paper is devoted to the following result.

Theorem 1. $CH^*(Gl(2n)/SO(2n)) \cong \mathbb{Z} \oplus \mathbb{Z}y$, where y is a codimension n cycle.

Throughout this paper $Gl(n)$, $O(n)$, $SO(n)$, etc. denote the complex algebraic groups of these types.

Theorem 1 should be contrasted with the following result which will be used in its proof.

Lemma 1. $CH^0(Gl(2n)/O(2n)) \cong \mathbb{Z}$ and $CH^i(Gl(2n)/O(2n)) = 0$ for $i \geq 1$.

Proof. $Gl(2n)/O(2n) \cong Symm_{2n}(\mathbb{C})$, where $Symm_{2n}(\mathbb{C})$ is the space of symmetric, non-degenerate $2n \times 2n$ matrices over \mathbb{C} . $Symm_{2n}(\mathbb{C})$ is an open subset of the vector space of all $2n \times 2n$ symmetric matrices over \mathbb{C} . By the fundamental exact sequence for Chow groups (Lemma 2 below), the Chow groups of any Zariski open subset of affine space vanish in codimensions higher than zero and are \mathbb{Z} in codimension zero. \square

There is much work in the literature on the geometry of symmetric spaces of the form G/K for G an adjoint group and K the fixed point set of an involution of G , see for example [DP]. However, these results do not include a computation of the Chow ring. Moreover, the Chow ring of such a G/K does not determine the Chow ring of the symmetric space G/K^0 where K^0 is the connected component of the identity in K . This can be significantly more complicated as Theorem 1 and Lemma 1 show.

In fact, there is currently no general theorem that computes the Chow ring of a reductive symmetric space G/K in terms of Lie theoretic data. The most

general result I know of is when G/K is a group; the Chow ring *mod* p is computed in Kac [K]. For a history of similar problems and extensive references to previous results, see the survey article [M].

In a subsequent paper, we will use Theorem 1 as the key step in the computation of the Chow ring of the classifying space $BSO(2n)$. The method of computing the Chow ring of $Gl(2n)/SO(2n)$ given in this paper can be extended to a computation of the cohomology of this symmetric space for all other cohomology theories.

Acknowledgements

I would like to thank Brendan Hassett, J. Peter May, Burt Totaro, Madhav Nori, and especially Ian Grojnowski, without whom this paper would be impossible to read if it ever appeared at all.

2 Basic results

For notation and conventions on Chow rings, we will refer to Fulton's *Intersection theory* [F] with one minor difference. The Chow ring that we will be using is the ring of algebraic cycles mod rational equivalence (denoted $CH^*(X)$) rather than the ring of cycles mod algebraic equivalence (denoted $A^*(X)$) as used by Fulton.

The basic result about Chow rings that we will need is the following.

Lemma 2. *Let $Y \subseteq X$ be a closed subvariety. Then*

$$CH_*Y \rightarrow CH_*X \rightarrow CH_*(X - Y) \rightarrow 0$$

is exact.

Recall that $CH^iX = CH_{\dim X - i}X$.

As in [FMSS], given an action of an algebraic group Γ on a variety X , one forms $CH_k^\Gamma X = Z_k^\Gamma X / R_k^\Gamma X$. Here $Z_k^\Gamma X$ is the free Abelian group generated by the Γ stable closed subvarieties of X and $R_k^\Gamma X$ is the subgroup generated by all divisors of eigenfunctions on Γ . A rational function f on X is an eigenfunction if $g \cdot f = \chi(g)f$ for all $g \in \Gamma$ and for some one dimensional character χ of Γ .

Theorem 2. *(Fulton, MacPherson, Sottile, Sturmfels)/[FMSS] If a connected solvable linear algebraic group Γ acts on a scheme X , then the canonical homomorphism $A_k^\Gamma X \rightarrow A_kX$ is an isomorphism.*

These results allow us to restrict attention to B -stable cycles in our proof of Theorem 1. In fact the decomposition of $Gl(2n)/SO(2n)$ into B -orbits is not used essentially in the proof of the theorem, but it does provide a convenient notation for cycles.

We now recall some results about $Gl(2n)/SO(2n)$ and its partition into B -orbits. All of these statements are well known and their proofs are easy linear algebra. We let $Symm_k(\mathbb{C})$ denote the subset of $Gl(k)$ consisting of symmetric matrices, and define the map

$$f : Gl(k) \rightarrow Symm_k(\mathbb{C})$$

by $f(g) = gg^t$. This map induces an isomorphism

$$Gl(k)/O(k) \cong Symm_k(\mathbb{C}),$$

and allows us to make the identification

$$Gl(k)/SO(k) \cong \{(q, \epsilon) | q \in Gl(k)/O(k) \text{ and } \epsilon^2 = \det(q)\}.$$

With this identification, the obvious double cover

$$\pi : Gl(2n)/SO(2n) \rightarrow Gl(2n)/O(2n)$$

takes a pair (q, ϵ) to q . We will identify $Symm_k(\mathbb{C})$ with the space of symmetric non-degenerate bilinear forms on \mathbb{C}^{2n} by setting $q_A(v, w) = v^t A w$, so that $Gl(2n)$ acts on bilinear forms by $(gq)(v, w) = q(g^t v, g^t w)$.

Define the Borel subgroup B to be the subgroup of upper triangular matrices, i.e. the stabilizer of the standard flag $F_1 \subset F_2 \subset \dots \subset F_{2n}$, where $F_i = \langle e_1, e_2, \dots, e_i \rangle$. We have an isomorphism between B -orbits in $Gl(2n)/O(2n)$ and involutions in the Weyl group S_{2n} :

$$B \backslash Gl(2n)/O(2n) \cong \{\omega \in S_{2n} \mid \omega^2 = 1\},$$

defined by sending a quadratic form q to the relative position between the standard flag F and the flag F^{\perp_q} of orthogonal complements with respect to q . It is well known and easy to see that this map is an isomorphism (this is a consequence of Gram-Schmidt orthonormalization).

To describe the B -orbits on $Gl(2n)/SO(2n)$, it is enough to describe the pullback of the orbits on $Gl(2n)/O(2n)$ through the double cover π . An inverse image $\pi^{-1}(O)$ is either a single orbit or a union of two disjoint orbits in the double cover $Gl(2n)/SO(2n)$.

Lemma 3. *The pullback of a B -orbit is the union of two disjoint orbits if and only if the permutation indexing the orbit is fixed point free.*

Let O be an orbit in $Gl(2n)/SO(2n)$ and $q \in O$. Then $\pi^{-1}(O)$ is a single B -orbit if and only if the stabilizer of q in B is disconnected; i.e. if and only if the stabilizer of q in the set of diagonal matrices T is disconnected. Choose q such that $q(x, y) = x^t \omega y$ where $\omega \in S_{2n}$ is the permutation indexing O . Then $q(e_i, e_j) = \delta_{\omega(i), j}$. The component group of the stabilizer of q in T is $(\mathbb{Z}/2\mathbb{Z})^l$, where l is the number of fixed points of ω on $\{1, 2, \dots, 2n\}$.

Definition 1. *A B -orbit O on $Gl(2n)/O(2n)$ is **fixed point free** if $\pi^{-1}(O)$ consists of two disjoint B -orbits.*

Proposition 1. *The Chow ring of $Gl(2n)/SO(2n)$ is generated by elements corresponding to closures of fixed point free orbits in $Gl(2n)/O(2n)$ along with the codimension zero cycle.*

The point of this paper is to prove that all but one of these codimension non-zero generators are rationally equivalent to zero.

Proof. As Lemma 1 states, the Chow ring of $Gl(2n)/O(2n)$ is trivial. Therefore, if O is a B -orbit in $Gl(2n)/O(2n)$ whose codimension is larger than zero, then $\pi^{-1}(O) \sim 0$ in $CH^*(Gl(2n)/SO(2n))$.

By Definition 1, any orbit in $Gl(2n)/O(2n)$ which is not fixed point free lifts to a single orbit in $Gl(2n)/SO(2n)$ and is therefore rationally equivalent to zero.

On the other hand, if an orbit O is fixed point free, then $\pi^{-1}(O) = O_+ \amalg O_-$ where O_+ and O_- are two disjoint copies of O distinguished by the sign of ϵ . Again, since the Chow ring of the base space is trivial, $0 \sim \pi^{-1}(O) \sim O_+ + O_-$, so $O_+ \sim -O_-$. Therefore, $CH^*(Gl(2n)/SO(2n))$ is generated by elements corresponding to closures of fixed point free orbits in $Gl(2n)/O(2n)$, along with the codimension zero cycle. \square

3 Proof of Theorem 1

Define the graph of B -orbits to be a graph whose vertices are B -orbits and whose edges are codimension 1 inclusion relations between orbit closures. This graph of B -orbits contains the full subgraph of fixed point free B -orbits. The proof of Theorem 1 was arrived at through a careful examination of the fixed point free graphs of $Gl(2n)/O(2n)$, for $n \leq 4$. These examples are reproduced in the appendix to make the proof easier to visualize.

The proof is by induction. We start by decomposing $Gl(2n)/SO(2n)$ into $2n$ disjoint subvarieties, each of which may be compared with $Gl(2n-2)/SO(2n-2)$. Our induction hypothesis will be $CH^*(Gl(2n-2)/SO(2n-2)) \cong \mathbb{Z}x_0 \oplus \mathbb{Z}y$, where x_0 is the codimension zero cycle and y is in codimension $n-1$. We will then construct a map from all but one of the $2n$ disjoint subvarieties to $Gl(2n-2)/SO(2n-2)$. This map will be a trivial fibration. The remaining subvariety is easily dealt with using Proposition 1. This map will take fixed point free orbits of $Gl(2n)/SO(2n)$ to fixed point free orbits of $Gl(2n-2)/SO(2n-2)$, hence the relevance of the examples in the appendix. We will use this map and the induction step to show that each of the $2n$ subsets contributes at most $\mathbb{Z} \oplus \mathbb{Z}$ to $CH^*(Gl(2n)/SO(2n))$. From there, a second and third induction and the results of the previous section will show that only two of these disjoint subsets actually contribute to the Chow ring.

We will start the induction with the case $n = 1$. Since there is only one fixed point free permutation in two letters, namely $(1\ 2)$, the Chow ring $CH^*(Gl(2)/SO(2))$ has a single generator in addition to the trivial codimension zero cycle. Since this single fixed point free orbit is the largest fixed point free orbit, it cannot be the zero of any B -semi-invariant function. Therefore, $CH^*(Gl(2)/SO(2)) \cong \mathbb{Z}x_0 \oplus \mathbb{Z}y$, where x_0 is the codimension zero cycle and y is this codimension 1 cycle.

We now define a decomposition of $Gl(2n)/SO(2n)$ into $2n$ disjoint subspaces.

Define X_i to be the subvariety of $Gl(2n)/SO(2n)$ consisting of pairs (q, ϵ) where $q(e_{2n}, e_{2n}) = q(e_{2n-1}, e_{2n}) = \cdots = q(e_{i+1}, e_{2n}) = 0$ and $q(e_i, e_{2n}) \neq 0$. The following properties of the varieties X_i are immediate.

Lemma 4.

$$Gl(2n)/SO(2n) = X_1 \amalg X_2 \amalg \cdots \amalg X_{2n}.$$

Also,

$$\overline{X_i} = \coprod_{j \geq i} X_j.$$

The X_i and the $\overline{X_i}$ are all quasi-projective varieties, and each X_i is B -stable.

Notice that if $(q, \epsilon) \in X_{2n}$, then the B -orbit through q is not fixed point free, and hence by Proposition 1, the only B -orbit in X_{2n} that contributes to the Chow ring is the codimension zero open orbit.

For $i < 2n$, we define a map

$$f_i : X_i \longrightarrow Gl(2n-2)/SO(2n-2)$$

as follows. For $(q, \epsilon) \in X_i$, we know that $q(e_{i+1}, e_{2n}) = q(e_{i+2}, e_{2n}) = \cdots = q(e_{2n}, e_{2n}) = 0$, while $q(e_i, e_{2n}) \neq 0$. Therefore, the quadratic form q is non-degenerate on $\langle e_i, e_{2n} \rangle$, the subspace of \mathbb{C}^{2n} generated by e_i and e_{2n} . The orthogonal complement $\langle e_i, e_{2n} \rangle^\perp$ with respect to q of this subspace is isomorphic to \mathbb{C}^{2n-2} .

Let q' be the quadratic form on \mathbb{C}^{2n-2} defined by restricting q to $\langle e_i, e_{2n} \rangle^\perp$ and let $\epsilon' = \epsilon/q(e_i, e_{2n})\sqrt{-1}$. As $\det(q) = -q(e_i, e_{2n})^2 \det(q')$, we know $(\epsilon')^2 = \det(q')$. Define $f_i(q, \epsilon) = (q', \epsilon')$.

This map takes B -orbits to \overline{B} -orbits, where \overline{B} is the stabilizer in $Gl(2n-2)$ of the standard flag associated to the basis

$$\overline{e_1}, \dots, \overline{e_{i-1}}, \overline{e_{i+1}}, \dots, \overline{e_{2n-1}},$$

where $\overline{e_j}$ is equal to the orthogonal projection of e_j onto $\langle e_i, e_{2n} \rangle^\perp$.

Lemma 5. f_i is a trivial fibration with fibers isomorphic to $\mathbb{C}^* \times \mathbb{C}^{2n+i-2}$.

Proof. Fix a quadratic form q' in $Gl(2n-2)/SO(2n-2)$. An element $q \in f^{-1}(q')$ is determined by the numbers

$$\begin{aligned} q(e_j, e_{2n}) & \text{ for } j < i, \text{ and} \\ q(e_j, e_i) & \text{ for } j \neq i. \end{aligned}$$

These numbers may be chosen freely subject only to the constraint that $q(e_i, e_{2n}) \neq 0$. \square

Lemma 6. $CH^*(X_i)$ is a quotient of $\mathbb{Z} \oplus \mathbb{Z}$.

Proof. Since the map f_i is a fibration with fibers isomorphic to an open subset of affine space, it induces a surjection of Chow rings ([F] example 1.9.2). By our induction hypothesis $CH^*(Gl(2n-2)/SO(2n-2)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

More specifically, the Chow ring of $Gl(2n-2)/SO(2n-2)$ is generated by cycles x_0 and y , where x_0 is the codimension zero cycle and y is in codimension $n-1$. Therefore, $CH^*(X_i)$ is generated by the pullbacks of these cycles. \square

Remark. In fact $CH^*(X_i) \cong \mathbb{Z} \oplus \mathbb{Z}$ as follows from the proof of Theorem 1 below. This can also easily be seen directly.

Proposition 2. *If $n > 1$, then $CH^*(Gl(2n)/SO(2n)) = CH^*(\overline{X_{2n}})$ is a quotient of $\mathbb{Z}\widetilde{x_0} \oplus \mathbb{Z}f_{2n-1}^*(y) \oplus \mathbb{Z}f_{2n-2}^*(y) \oplus \cdots \oplus \mathbb{Z}f_1^*(y)$, where $\widetilde{x_0}$ is the codimension zero cycle in $\overline{X_{2n}}$.*

Proof. We will show by induction on i that $CH^*(\overline{X_i})$ is a quotient of $\mathbb{Z}f_i^*(y) \oplus \mathbb{Z}f_{i-1}^*(y) \oplus \cdots \oplus \mathbb{Z}f_1^*(y)$.

For the case $i = 1$, since $\overline{X_1} = X_1$, Lemma 6 tells us that $CH^*(\overline{X_1})$ is a quotient of $\mathbb{Z}f_1^*(x_0) \oplus \mathbb{Z}f_1^*(y)$. The pullback of the codimension zero orbit $f_1^*(x_0)$ is indexed by the permutation $(1 \ 2n)$ so since $n > 1$ it is not a fixed point free orbit in $Gl(2n)/SO(2n)$. Therefore, by Proposition 1, $f_1^*(x_0)$ is rationally equivalent to zero in $CH^*(Gl(2n)/SO(2n))$, and the subvariety $\overline{X_1}$ contributes at most $f_1^*(y)$ to the Chow ring of the whole symmetric space. This completes the $i = 1$ case.

As stated in Lemma 2, we know that

$$CH^*(\overline{X_{i-1}}) \rightarrow CH^*(\overline{X_i}) \rightarrow CH^*(X_i) \rightarrow 0$$

is exact. Therefore, by induction, $CH^*(\overline{X_i})$ is a quotient of $\mathbb{Z}f_i^*(x_0) \oplus \mathbb{Z}f_i^*(y) \oplus \mathbb{Z}f_{i-1}^*(y) \oplus \cdots \oplus \mathbb{Z}f_1^*(y)$. Again, we note that $f_i^*(x_0)$ is indexed by the permutation $(i \ 2n)$ so is not fixed point free. For $i < 2n$, this orbit has codimension larger than zero, so by Proposition 1, this orbit is rationally equivalent to zero in $CH^*(Gl(2n)/SO(2n))$. Finally, if $i = 2n$, as we noticed previously (immediately following Lemma 4), the subvariety X_{2n} contributes only the codimension zero cycle $\widetilde{x_0}$. \square

Note that we have reduced the generators down to the largest fixed point free B -orbit in each of the X_i . These $2n - 1$ B -orbits are visible in the appendix along the bottom right hand side of each graph.

We claim that $f_j^*(y)$ is rationally equivalent to zero in $CH^*(Gl(2n)/SO(2n))$ for $j < 2n$. To this end, we inductively show that $CH^*(\overline{X_j})$ is a quotient of $\mathbb{Z}f_j^*(y)$ for $j < 2n$.

Define the function $g_j : \overline{X_j} \rightarrow \mathbb{C}$ by $g_j(q) = q(e_j, e_{2n})$. This function is clearly non-zero on X_j and is clearly zero on all of $\overline{X_{j-1}}$. To see that this zero is a simple one, we look at the function restricted to a line in $Gl(2n)/SO(2n)$. Let $q \in X_j, q' \in X_{j-1}$ and consider the line of quadratic forms $\{q' + aq \mid a \in \mathbb{C}\}$. Clearly the function g_j has a simple zero along this line. This line meets X_{j-1} only when $a = 0$ and is otherwise completely contained in X_j . It is clear that the line intersects $\overline{X_{j-1}}$ transversally since $\overline{X_{j-1}}$ is an open subvariety of the vector space of quadratic forms for which $q(e_j, e_{2n}) = 0$. Therefore, the intersection is transversal and the zero is simple.

By our induction step, $CH^*(\overline{X_{j-1}})$ is a quotient of $\mathbb{Z}f_{j-1}^*(y)$, so $CH^*(\overline{X_j})$ is a quotient of $\mathbb{Z}f_{j-1}^*(y) \oplus \mathbb{Z}f_j^*(y)$. Let γ be the B -orbit in $Gl(2n-2)/SO(2n-2)$ representing the cycle y . Restricting g_j to the closure of $f_j^{-1}(\gamma)$ in $\overline{X_j}$, this argument shows that $f_{j-1}^*(y) \sim 0$ in $CH^*(\overline{X_j})$.

We have shown $f_j^*(y) \sim 0$ for $j < 2n - 1$. Consider $f_{2n-1}^*(y)$. This cycle is the closure of the largest fixed point free orbit, so any B -semi-invariant function vanishing along it must be defined on the closure of a non-fixed point free orbit. Therefore, any such function will produce a relation involving both copies of the orbit pulled back from $Gl(2n)/O(2n)$. Since we already know that the sum of these copies is rationally equivalent to zero, such a function can produce no new relations. The same argument shows that no multiple of $f_{2n-1}^*(y)$ is rationally equivalent to zero.

We have shown that $f_j^*(y) \sim 0$ for $1 < j < 2n - 1$ and that no multiple of $f_{2n-1}^*(y)$ is rationally equivalent to zero. This combined with Proposition 2 proves Theorem 1.

4 Appendix

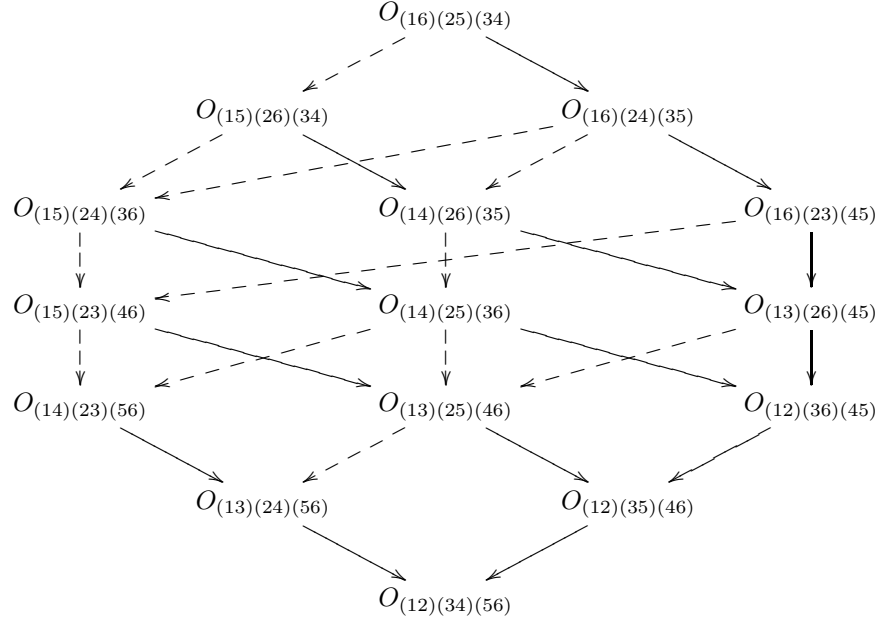
As mentioned in the proof, the graph of fixed point free B -orbits for $Gl(2)/SO(2)$ is a single point corresponding to the orbit $O_{(12)}$.

The graph of fixed point free B -orbits for $Gl(4)/SO(4)$ is:

$$\begin{array}{c} O_{(14)(23)} \\ \downarrow \\ O_{(13)(24)} \\ \downarrow \\ O_{(12)(34)} \end{array}$$

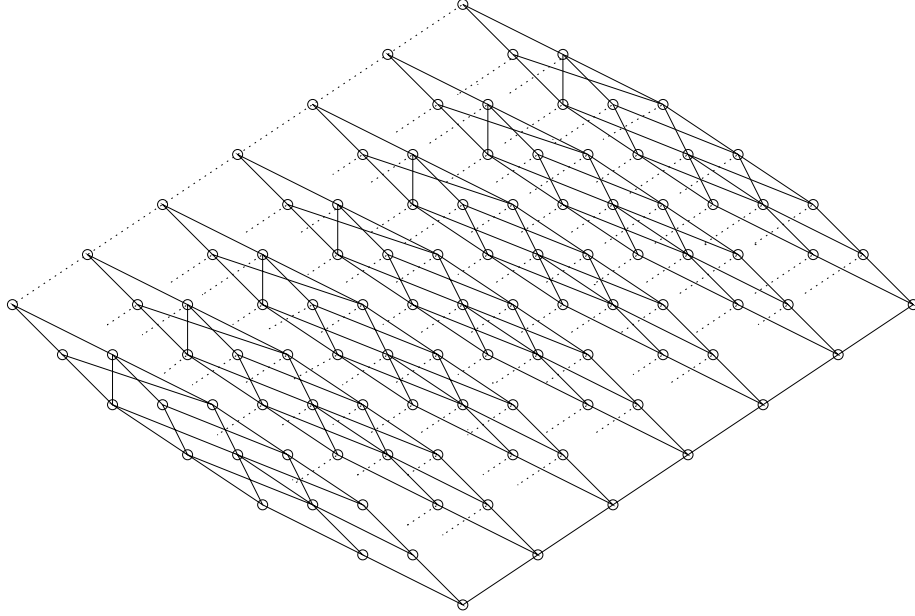
where the codimensions of the orbits are (counting from the bottom) 2, 3, and 4 and the arrows represent inclusion of an orbit in the closure of the larger orbit. Since the graph for $Gl(2)/SO(2)$ is a single point, this is also its decomposition into subgraphs.

This leads to the induction step for the graph of fixed point free B -orbits for $Gl(6)/SO(6)$, namely:



where the solid diagonal lines that are more or less parallel to each other represent inclusion relations within the X_i and the dotted lines represent other inclusion relations that are not relevant to our proof. Furthermore, orbits that appear on the same row have the same codimension.

The following is the graph of fixed point free B -orbits in $Gl(8)/O(8)$. Notice its seven subgraphs.



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